

**Variance derivatives and estimating realised  
variance from high-frequency data**

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- We denote the initial time (today) by  $t_0 \equiv 0$ . We consider a stock whose price, at time  $t$ , is  $S(t)$ . We consider a time interval  $[t_0, T]$  which is partitioned into  $N$  time periods (not necessarily equal in length) whose end-points are  $t_j$ ,  $j = 1, 2, \dots, N$ , where  $0 \equiv t_0 < t_1 < \dots < t_{j-1} < t_j < \dots < t_N \equiv T$ .
- What difference does it make if realised variance is measured by log changes squared (i.e.  $\sum_{i=1}^N (\log(S(t_i)/S(t_{i-1})))^2$ ) or by proportional differences squared (i.e.  $\sum_{i=1}^N ((S(t_i)/S(t_{i-1})) - 1)^2$ )?
- What impact does monitoring frequency (i.e. the value of  $N$  above) have on the measurement of realised variance?
- What impact do jumps in the underlying stock price have on the measurement of realised variance?
- Building on Broadie and Jain (2008), Carr and Lee (2009) and Hong (2004), we will try to answer these questions.

- Our results have two important applications:
- 1./ The pricing (under an equivalent martingale measure (EMM)  $\mathbb{Q}$ ) of variance swaps which pay  $\sum_{i=1}^N (\log(S(t_i)/S(t_{i-1})))^2$  (which is how the payoffs are usually defined in practice) and of proportional variance swaps which pay  $\sum_{i=1}^N ((S(t_i)/S(t_{i-1})) - 1)^2$  at maturity  $T$ . In particular, we consider the case when  $N$  is infinite (continuously monitored) and the case when  $N$  is finite (discretely monitored - as they must always be in practice).
- 2./ Given observations of  $S(t_i)$  for times  $t_i$ ,  $i = 1, 2, \dots, N$  (from historical data under the real-world physical measure  $\mathbb{P}$ ), what can we say about the process which generated this data? We are thinking, in particular, of high-frequency data (at least several, perhaps, a few hundred observations per day).

- For the first two-thirds of my talk, I will focus on variance swaps and model stock price dynamics under an EMM  $\mathbb{Q}$ .
- Nearly all papers on variance swaps have focussed on the log-contract replication approach (eg. Neuberger (1990), Dupire (1993), Derman et al. (1999)).
- However, there is a completely different approach (see Hong (2004) and Broadie and Jain (2008)) which utilises characteristic functions. We build upon this approach. However, firstly, we discuss the assumed stock price dynamics.

- We construct the stock price process by assuming that the log of the stock price is a time-changed Lévy process (allows a very generic process which includes (nearly) all models seen in the literature).
- We have a Lévy process (eg Brownian motion, Kou (2002) jump-diffusion, Variance Gamma or CGMY) denoted by  $X_t$ , satisfying  $X_{t_0} = 0$ . We assume that we mean-correct  $X_t$  so that  $\exp(X_t)$  is a (non-constant) martingale (under  $\mathbb{Q}$ ) - with respect to the natural filtration generated by  $X_t$  i.e. that  $\mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(X_t)] = \exp(X_{t_0}) = 1$  for all  $t \geq t_0$ .
- Lévy-Khinchin formula implies we can write the (mean-corrected) characteristic exponent  $\bar{\psi}_X(z)$  (defined via  $\mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(iuX_t)] \equiv \exp(-(t - t_0)\bar{\psi}_X(u))$ ) in the form:

$$-\bar{\psi}_X(z) = -\frac{1}{2}\sigma^2(z^2 + iz) + \int_{-\infty}^{\infty} (\exp(izx) - 1 - iz(\exp(x) - 1))\nu(dx).$$

For future reference, ' denotes differentiation i.e.  $\bar{\psi}'_X(z) \equiv \partial\bar{\psi}_X(z)/\partial z$ ,  $\bar{\psi}''_X(z) \equiv \partial^2\bar{\psi}_X(z)/\partial z^2$  and  $\bar{\psi}'''_X(z) \equiv \partial^3\bar{\psi}_X(z)/\partial z^3$ .

- For the case of Brownian motion, “ $X_t = -\frac{1}{2}\sigma^2t + \sigma W(t)$  where  $W(t)$  is standard (driftless) Brownian motion”.

- We assume that we have a non-decreasing, continuous time-change process denoted by  $Y_t$ . We normalise so that  $Y_{t_0} = t_0 \equiv 0$ .
- In general,  $Y_t$  may be correlated with  $X_t$ .
- Our assumption, for example, allows  $Y_t$  to be of the form  $Y_t = \int_{t_0}^t y_s ds$  where the activity rate  $y_t$  (which must be non-negative) follows, for example, a Heston (1993) square-root process, a non-Gaussian OU process (Barndorff-Nielsen and Shephard (2001)) or it could follow the Heston (1993) plus jumps process of Duffie et al. (2000). In the latter two cases,  $y_t$  is discontinuous but  $Y_t$  is always continuous.
- (The time-change will allow us to model stochastic volatility / leverage / volatility clustering type effects).

- We time-change the Lévy process  $X_t$  by  $Y_t$  to get a process  $X_{Y_t}$ , with  $X_{Y_{t_0}} = 0$ .
- The stock price  $S(t)$ , at time  $t$ , is assumed to have the following dynamics (under  $\mathbb{Q}$ ):

$$S(t) = S(t_0) \exp\left(\int_{t_0}^t (r(s) - q(s))ds + X_{Y_t}\right).$$

- Here,  $r(t)$  is the risk-free interest-rate and  $q(t)$  is the dividend yield (assumed finite and deterministic), at time  $t$ .
- To lighten notation, I will henceforth write equations as if  $r(t) - q(t) \equiv 0$  for all  $t$  (or equivalently work with forward or future prices - the paper considers the general case). Hence,  $S(t) = S(t_0) \exp(X_{Y_t})$ .



- We now define, for all  $t \geq t_0$ :

$$\Xi_t(u) \equiv \exp(iuX_{Y_t} + Y_t\bar{\psi}_X(u)).$$

Since the mean-corrected characteristic exponent  $\bar{\psi}_X(u)$  is defined via:

$\mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(iuX_t)] = \exp(-(t - t_0)\bar{\psi}_X(u))$ , then  $\exp(iuX_t + (t - t_0)\bar{\psi}_X(u))$  is a martingale, under  $\mathbb{Q}$ , with respect to the natural filtration generated by  $X_t$ .

- By a “randomising time” (Optional Stopping Theorem) argument, for any  $u$ ,  $\Xi_t(u)$  is a martingale, under  $\mathbb{Q}$ , with respect to the filtration generated by  $\mathcal{F}_t \equiv \{X_t \cup Y_t\}$ .
- In particular,

$$\mathbb{E}_{t_{j-1}}^{\mathbb{Q}}\left[\frac{\Xi_{t_j}(u)}{\Xi_{t_{j-1}}(u)}\right] = \mathbb{E}_{t_{j-1}}^{\mathbb{Q}}[\exp(iu(X_{Y_{t_j}} - X_{Y_{t_{j-1}}}) + (Y_{t_j} - Y_{t_{j-1}})\bar{\psi}_X(u))] = 1.$$

- We now introduce what we call the joint extended characteristic function  $\Phi(z; j)$ , which we define, for each  $j$ ,  $j = 1, \dots, N$ , by:

$$\begin{aligned}
 \Phi(z; j) &\equiv \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\exp\left(iz \log \frac{S(t_j)}{S(t_{j-1})}\right)\right] = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\exp\left(iz(X_{Y_{t_j}} - X_{Y_{t_{j-1}}})\right)\right] \\
 &= \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\exp\left(iz(X_{Y_{t_j}} - X_{Y_{t_{j-1}}}) + (Y_{t_j} - Y_{t_{j-1}})\bar{\psi}_X(z)\right) \exp\left(-(Y_{t_j} - Y_{t_{j-1}})\bar{\psi}_X(z)\right)\right] \\
 &= \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\mathbb{E}_{t_{j-1}}^{\mathbb{Q}}\left[\frac{\Xi_{t_j}(z)}{\Xi_{t_{j-1}}(z)} \exp\left(-(Y_{t_j} - Y_{t_{j-1}})\bar{\psi}_X(z)\right)\right]\right].
 \end{aligned}$$

- (Note as an aside,  $\Phi(z; j)$  is “a kind of forward characteristic function”. One can compute  $\Phi(z; j)$ , for cases of interest, via conditioning arguments and by using results in Carr and Wu (2004) and Duffie et al. (2000), so we will say nothing more about this.)

- We note that the joint extended characteristic function  $\Phi(z; j)$  allows us to immediately evaluate the price of a discretely monitored proportional variance swap. We let  $iz = 2$  in the equation for  $\Phi(z; j)$ , then sum over  $j$  and simplify.
- $\Rightarrow$ : The price  $PVS(t_0, T, N)$ , at time  $t_0$ , of a (discretely monitored) proportional variance swap (paying  $\sum_{i=1}^N ((S(t_i)/S(t_{i-1})) - 1)^2$  at time  $T$ ) is:

$$PVS(t_0, T, N) = P(t_0, T) \left( \sum_{j=1}^N (\Phi(-2i; j) - 1) \right).$$

Here,  $P(t_0, T)$  is the price of a zero-coupon bond, at time  $t_0$ , that matures at time  $T$ .

- We will examine the limit as  $N \rightarrow \infty$  of this equation later.

- Now we differentiate  $\Phi(z; j)$  with respect to  $z$  and divide by  $i$ :

$$\begin{aligned} \frac{1}{i} \frac{\partial \Phi(z; j)}{\partial z} &= \mathbb{E}_{t_0}^{\mathbb{Q}} \left[ \log \frac{S(t_j)}{S(t_{j-1})} \exp\left(iz \log \frac{S(t_j)}{S(t_{j-1})}\right) \right] \\ &= \mathbb{E}_{t_0}^{\mathbb{Q}} \left[ \mathbb{E}_{t_{j-1}}^{\mathbb{Q}} \left[ \frac{\Xi_{t_j}(z)}{\Xi_{t_{j-1}}(z)} \exp\left(-(Y_{t_j} - Y_{t_{j-1}}) \bar{\psi}_X(z)\right) \right. \right. \\ &\quad \left. \left. \left( \varpi^{(j)}(iz) + ((X_{Y_{t_j}} - X_{Y_{t_{j-1}}}) - i \bar{\psi}'_X(z)(Y_{t_j} - Y_{t_{j-1}})) \right) \right] \right], \quad \text{where} \\ \varpi^{(j)}(iz) &\equiv i \bar{\psi}'_X(z)(Y_{t_j} - Y_{t_{j-1}}). \end{aligned}$$

- It is now straightforward to value log-forward-contracts (paying  $\log(S(t_N)/S(t_0))$  at time  $T$ ). We set  $iz = 0$ , then we sum from  $j = 1$  to  $N$  and then simplify. The price  $\text{LFC}(t_0, T)$ , at time  $t_0$ , of a log-forward-contract is:

$$\text{LFC}(t_0, T) = P(t_0, T) i \bar{\psi}'_X(0) \mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T - Y_{t_0}] \equiv P(t_0, T) m_X \mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T - Y_{t_0}].$$

Note  $m_X$  defined by  $m_X \equiv i \bar{\psi}'_X(0)$  is real.

- We differentiate again with respect to  $z$  and again divide by  $i$ :

$$\begin{aligned}
-\frac{\partial^2 \Phi(z; j)}{\partial z^2} &= \mathbb{E}_{t_0}^{\mathbb{Q}} \left[ \left( \log \frac{S(t_j)}{S(t_{j-1})} \right)^2 \exp \left( iz \log \frac{S(t_j)}{S(t_{j-1})} \right) \right] \\
&= \mathbb{E}_{t_0}^{\mathbb{Q}} \left[ \mathbb{E}_{t_{j-1}}^{\mathbb{Q}} \left[ \frac{\bar{\Xi}_{t_j}(z)}{\bar{\Xi}_{t_{j-1}}(z)} \exp \left( -(Y_{t_j} - Y_{t_{j-1}}) \bar{\psi}_X(z) \right) \right. \right. \\
&\quad \left. \left( \varpi^{(j)2}(iz) \right. \right. \\
&\quad \left. \left. + \left\{ 2\varpi^{(j)}(iz) \left( (X_{Y_{t_j}} - X_{Y_{t_{j-1}}}) - i\bar{\psi}'_X(z)(Y_{t_j} - Y_{t_{j-1}}) \right) \right\} \right. \right. \\
&\quad \left. \left. + \left( (X_{Y_{t_j}} - X_{Y_{t_{j-1}}}) - i\bar{\psi}'_X(z)(Y_{t_j} - Y_{t_{j-1}}) \right)^2 - \bar{\psi}''_X(z)(Y_{t_j} - Y_{t_{j-1}}) \right. \right. \\
&\quad \left. \left. + \bar{\psi}''_X(z)(Y_{t_j} - Y_{t_{j-1}}) \right) \right] \right].
\end{aligned}$$

- The price, at time  $t_0$ , of a variance swap  $VS(t_0, T, N)$  can be obtained by setting  $iz = 0$ , summing from  $j = 1$  to  $N$  and simplifying: The price  $VS(t_0, T, N)$  is:

$$\begin{aligned}
 & VS(t_0, T, N) \\
 &= P(t_0, T) \mathbb{E}_{t_0}^{\mathbb{Q}} \left[ \sum_{j=1}^N \mathbb{E}_{t_{j-1}}^{\mathbb{Q}} [\varpi^{(j)2}(0)] \right] \\
 &+ P(t_0, T) \mathbb{E}_{t_0}^{\mathbb{Q}} \left[ \sum_{j=1}^N \mathbb{E}_{t_{j-1}}^{\mathbb{Q}} [2m_X(Y_{t_j} - Y_{t_{j-1}}) ((X_{Y_{t_j}} - X_{Y_{t_{j-1}}}) - m_X(Y_{t_j} - Y_{t_{j-1}}))] \right] \\
 &+ P(t_0, T) \bar{\psi}_X''(0) \mathbb{E}_{t_0}^{\mathbb{Q}} \left[ \sum_{j=1}^N (Y_{t_j} - Y_{t_{j-1}}) \right]. \tag{1}
 \end{aligned}$$

- Note that  $\varpi^{(j)}(0)$  is the drift of log of the stock price (over the time interval  $t_{j-1}$  to  $t_j$ ) (it is real and for Brownian motion and a deterministic time-change it is “ $(r - q - \frac{1}{2}\sigma^2)(t_j - t_{j-1})$ ”).
- Here  $m_X \equiv i\bar{\psi}_X'(0)$  (note  $m_X$  is real and for Brownian motion it is “ $-\frac{1}{2}\sigma^2$ ”).
- Lets look at each of the three lines of equation (1) in turn.

- Again,  $VS(t_0, T, N)$

$$\begin{aligned}
&= P(t_0, T) \mathbb{E}_{t_0}^{\mathbb{Q}} \left[ \sum_{j=1}^N \mathbb{E}_{t_{j-1}}^{\mathbb{Q}} [\varpi^{(j)2}(0)] \right] \\
&+ P(t_0, T) \mathbb{E}_{t_0}^{\mathbb{Q}} \left[ \sum_{j=1}^N \mathbb{E}_{t_{j-1}}^{\mathbb{Q}} [2m_X(Y_{t_j} - Y_{t_{j-1}}) ((X_{Y_{t_j}} - X_{Y_{t_{j-1}}}) - m_X(Y_{t_j} - Y_{t_{j-1}}))] \right] \\
&+ P(t_0, T) \bar{\psi}_X''(0) \mathbb{E}_{t_0}^{\mathbb{Q}} \left[ \sum_{j=1}^N (Y_{t_j} - Y_{t_{j-1}}) \right].
\end{aligned}$$

- Note that, with a deterministic time-change,  $\varpi^{(j)2}(0)$  is  $O(1/N^2)$ . Broadie and Jain (2008) show that it is  $O(1/N^2)$  if the activity-rate of the time-change is Heston (1993). In the paper, we show that it is  $O(1/N^2)$  for “almost any” time-change.
- Hence the first line is  $O(1/N)$  and  $\rightarrow 0$  as  $N \rightarrow \infty$ .
- Since  $\varpi^{(j)}(0)$  is real,  $\varpi^{(j)2}(0)$  is definitely non-negative and zero only if the drift of the log of the stock price is identically equal to zero.

- Again, the second line is:

$$P(t_0, T) \mathbb{E}_{t_0}^{\mathbb{Q}} \left[ \sum_{j=1}^N \mathbb{E}_{t_{j-1}}^{\mathbb{Q}} \left[ 2m_X(Y_{t_j} - Y_{t_{j-1}}) \left( (X_{Y_{t_j}} - X_{Y_{t_{j-1}}}) - m_X(Y_{t_j} - Y_{t_{j-1}}) \right) \right] \right].$$

- Note  $\mathbb{E}_{t_{j-1}}^{\mathbb{Q}} \left[ (X_{Y_{t_j}} - X_{Y_{t_{j-1}}}) - m_X(Y_{t_j} - Y_{t_{j-1}}) \right] \equiv 0$  (by construction it is a martingale eg the whole term is standard Brownian motion).
- Therefore, if  $X_t$  and  $Y_t$  are independent, the second line is identically equal to zero.
- $m_X$  is always negative (eg for Brownian motion it is “ $-\frac{1}{2}\sigma^2$ ”). Therefore, if  $X_t$  and  $Y_t$  are negatively correlated, the second term is positive.
- Results in Broadie and Jain (2008) show, for Heston (1993) that the (absolute value of the) second line is  $O(1/N)$ . In the paper, we show that it is  $O(1/N)$  for any Lévy process and “almost any” time-change.



- Again,  $VS(t_0, T, N)$

$$\begin{aligned}
&= P(t_0, T) \mathbb{E}_{t_0}^{\mathbb{Q}} \left[ \sum_{j=1}^N \mathbb{E}_{t_{j-1}}^{\mathbb{Q}} [\varpi^{(j)2}(0)] \right] \\
&+ P(t_0, T) \mathbb{E}_{t_0}^{\mathbb{Q}} \left[ \sum_{j=1}^N \mathbb{E}_{t_{j-1}}^{\mathbb{Q}} [2m_X(Y_{t_j} - Y_{t_{j-1}}) ((X_{Y_{t_j}} - X_{Y_{t_{j-1}}}) - m_X(Y_{t_j} - Y_{t_{j-1}}))] \right] \\
&+ P(t_0, T) \bar{\psi}_X''(0) \mathbb{E}_{t_0}^{\mathbb{Q}} [Y_T - Y_{t_0}].
\end{aligned}$$

- The term  $\mathbb{E}_{t_0}^{\mathbb{Q}} [\sum_{j=1}^N (Y_{t_j} - Y_{t_{j-1}})] = \mathbb{E}_{t_0}^{\mathbb{Q}} [Y_T - Y_{t_0}]$  due to a telescoping sum.
- The third line is the price of the continuously monitored version of the variance swap.

- Again,  $VS(t_0, T, N)$

$$\begin{aligned}
&= P(t_0, T) \mathbb{E}_{t_0}^{\mathbb{Q}} \left[ \sum_{j=1}^N \mathbb{E}_{t_{j-1}}^{\mathbb{Q}} [\varpi^{(j)2}(0)] \right] \\
&+ P(t_0, T) \mathbb{E}_{t_0}^{\mathbb{Q}} \left[ \sum_{j=1}^N \mathbb{E}_{t_{j-1}}^{\mathbb{Q}} [2m_X(Y_{t_j} - Y_{t_{j-1}}) ((X_{Y_{t_j}} - X_{Y_{t_{j-1}}}) - m_X(Y_{t_j} - Y_{t_{j-1}}))] \right] \\
&+ P(t_0, T) \bar{\psi}_X''(0) \mathbb{E}_{t_0}^{\mathbb{Q}} [Y_T - Y_{t_0}].
\end{aligned}$$

- The price of a (discretely monitored) variance swap is the sum of three terms: A non-negative “drift-related” term, a “covariance” term which is non-negative (respectively, zero) if  $\text{Correl}(X_t, Y_t)$  is negative (respectively, zero) and the price of the continuously monitored version of the variance swap.
- In particular, if the “covariance” term is non-positive, a discretely monitored variance swap is always worth than its continuously monitored counterpart.
- Convergence is always  $O(1/N)$ .

- We saw earlier that the price  $PVS(t_0, T, N)$ , at time  $t_0$ , of a (discretely monitored) proportional variance swap (paying  $\sum_{i=1}^N ((S(t_i)/S(t_{i-1})) - 1)^2$  at time  $T$ ) is:

$$PVS(t_0, T, N) = P(t_0, T) \left( \sum_{j=1}^N (\Phi(-2i; j) - 1) \right).$$

- Hence:

$$\begin{aligned} \lim_{N \rightarrow \infty} PVS(t_0, T, N) &= \lim_{N \rightarrow \infty} P(t_0, T) \left( \sum_{j=1}^N (\Phi(-2i; j) - 1) \right) \\ &= P(t_0, T) \lim_{N \rightarrow \infty} \sum_{j=1}^N \mathbb{E}_{t_0}^{\mathbb{Q}} \left[ \frac{\Xi_{t_j}(-2i)}{\Xi_{t_{j-1}}(-2i)} (\exp(-(Y_{t_j} - Y_{t_{j-1}}) \bar{\psi}_X(-2i)) - 1) \right] \\ &= -P(t_0, T) \bar{\psi}_X(-2i) \mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T - Y_{t_0}] + O(1/N). \end{aligned}$$

- Hence, the price of the continuously monitored version of the proportional variance swap is  $-P(t_0, T) \bar{\psi}_X(-2i) \mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T - Y_{t_0}]$ .
- Convergence is also  $O(1/N)$ .

- From the previous slide,

$$\text{PVS}(t_0, T, N) = P(t_0, T) \left( \sum_{j=1}^N (\Phi(-2i; j) - 1) \right) \text{ with}$$

$$\Phi(-2i; j) = \mathbb{E}_{t_0}^{\mathbb{Q}} \left[ \frac{\Xi_{t_j}(-2i)}{\Xi_{t_{j-1}}(-2i)} \exp(-(Y_{t_j} - Y_{t_{j-1}}) \bar{\psi}_X(-2i)) \right].$$

Hence, it is clear (since  $\bar{\psi}_X^{(k)}(-2i) < 0$  eg. for Brownian motion  $\bar{\psi}_X^{(k)}(-2i) = -\sigma^2$ ) that when  $X_t$  and  $Y_t$  are positively correlated then the price of a discretely monitored proportional variance swap is higher than the price of the same discretely monitored proportional variance swap under the assumption that they are independent (the opposite way round to a variance swap).

- Under the assumption of independence, a discretely monitored proportional variance swap is always worth at least as much as an otherwise identical continuously monitored proportional variance swap (the same way round as a variance swap).

- We have explicit expressions for the prices of variance swaps and proportional variance swaps (both discretely monitored and continuously monitored). Discretely monitored prices tend to their continuously monitored counterparts as  $O(1/N)$  (for both variance swaps and proportional variance swaps).
- In the paper, we prove  $O(1/N)$  convergence is also true for discontinuous time-changes.
- In the paper, we prove  $O(1/N)$  convergence is also true for gamma swaps, self-quantoeed variance swaps and skewness swaps.
- The prices of continuously monitored variance swaps and proportional variance swaps (and also gamma swaps and skewness swaps) do **NOT** depend upon  $\text{Correl}(X_t, Y_t)$ .
- Can easily see dependence of discretely monitored versions of these swaps on  $\text{Correl}(X_t, Y_t)$ .
- In particular,

$$\text{VS}(t_0, T, N) \geq \text{VS}(t_0, T, \infty) \text{ provided } \text{Correl}(X_t, Y_t) \leq 0,$$

(and a non-positive correlation seems most likely in practice).

- The price of a continuously monitored proportional variance swap is:

$$\text{PVS}(t_0, T, \infty) = -P(t_0, T)\bar{\psi}_X(-2i)\mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T - Y_{t_0}].$$

- The price of a continuously monitored variance swap is:

$$\text{VS}(t_0, T, \infty) = P(t_0, T)\bar{\psi}_X''(0)\mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T - Y_{t_0}].$$

- The price of a log-forward-contract is:

$$\text{LFC}(t_0, T) = P(t_0, T)i\bar{\psi}_X'(0)\mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T - Y_{t_0}] \equiv P(t_0, T)m_X\mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T - Y_{t_0}].$$

- Hence:

$$\frac{\text{VS}(t_0, T, \infty)}{\text{LFC}(t_0, T)} = \frac{\bar{\psi}_X''(0)}{m_X}, \quad \frac{\text{PVS}(t_0, T, \infty)}{\text{LFC}(t_0, T)} = \frac{-\bar{\psi}_X(-2i)}{m_X}.$$

Carr and Lee (2009) have already proven the left-hand-side equation (i.e. for variance swaps (VS)) by a different method. In the paper, we show similar analogous results, not only for proportional variance swaps, but also for other types of variance derivatives.

- Hence, given vanilla prices, can price variance swaps and proportional variance swaps independent of any assumption on  $Y_t$  (and therefore robust to model (mis-)specification).

- For the case, when  $X_t$  is Brownian motion with volatility  $\sigma$ :

We have:  $\bar{\psi}_X(z) = \sigma^2(z^2 + iz)/2$ ,  $m_X = -\sigma^2/2$ ,  $\bar{\psi}_X''(0) = \sigma^2$ ,  $\bar{\psi}_X''(-i) = \sigma^2$ ,  $\bar{\psi}_X'''(0) = 0$  and  $\bar{\psi}_X(-2i) = -\sigma^2$ .

$$\frac{\text{VS}(t_0, T, \infty)}{-\text{LFC}(t_0, T)} = 2, \quad \frac{\text{PVS}(t_0, T, \infty)}{-\text{LFC}(t_0, T)} = 2.$$

- The left-hand-side equation restates Neuberger (1990), Dupire (1993) and Derman et al. (1999):

The price of a variance swap equals (minus) two times the price of a log-forward-contract (with the assumption of continuous sample paths (i.e. the log of the stock price is time-changed Brownian motion)).

- The right-hand-side equation says that it makes **no difference** if realised variance is measured by log changes squared (i.e.  $\sum_{i=1}^N (\log(S(t_i)/S(t_{i-1})))^2$ ) or by proportional differences squared (i.e.  $\sum_{i=1}^N ((S(t_i)/S(t_{i-1})) - 1)^2$ ) **when there are no jumps (i.e. continuous sample paths) and when  $N = \infty$  (i.e. continuously monitored)**.

- For the case, when  $X_t$  is a compound Poisson process with a fixed jump amplitude  $a$  (and with no diffusion component), then we have:

$$\frac{\text{VS}(t_0, T, \infty)}{-\text{LFC}(t_0, T)} = \frac{a^2}{(\exp(a) - 1 - a)} \approx 2 \left(1 - \frac{a}{3}\right),$$

$$\frac{\text{PVS}(t_0, T, \infty)}{-\text{LFC}(t_0, T)} = \frac{(\exp(a) - 1)^2}{(\exp(a) - 1 - a)} \approx 2 \left(1 + \frac{2a}{3}\right),$$

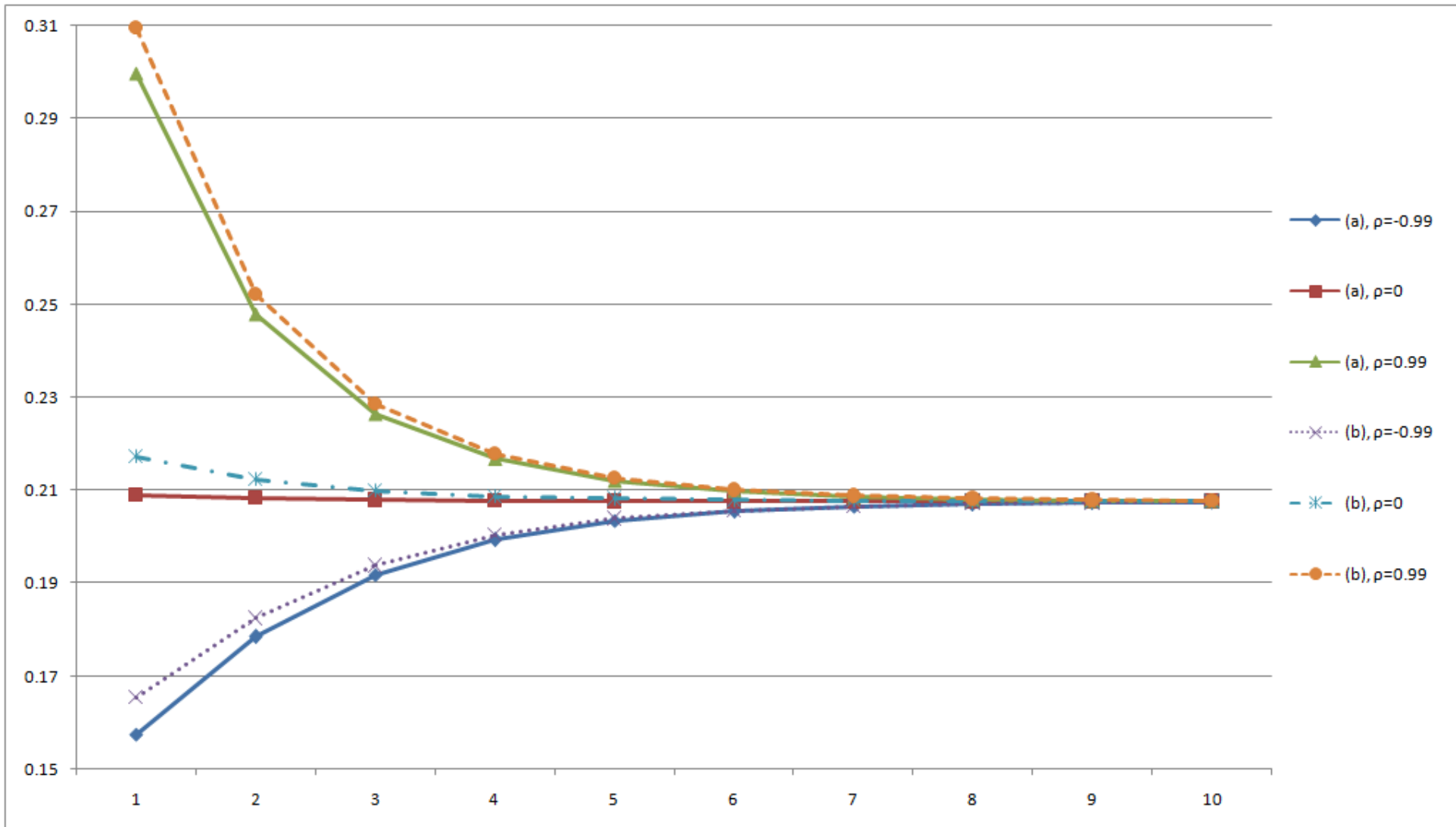
where, in each part, the first term is exact and the second term is the expansion of the first term to leading order when  $|a|$  is small.

- $\Rightarrow$ : The prices of variance swaps and proportional variance swaps have the **opposite sensitivities** to jumps (and the impact will be larger in magnitude (perhaps, twice as large) for proportional variance swaps).
- The right-hand-side equation suggests that it will make a **big difference** if realised variance is measured by log changes squared (i.e.  $\sum_{i=1}^N (\log(S(t_i)/S(t_{i-1})))^2$ ) or by proportional differences squared (i.e.  $\sum_{i=1}^N ((S(t_i)/S(t_{i-1})) - 1)^2$ ) **when there are (large) jumps**.



- We now consider some numerical examples.
- We consider variance swaps and proportional variance swaps, with maturity  $T = 0.5$ , and with  $N$  (equally-spaced) monitoring times where  $N = 2^{(J-1)}$ , for  $J = 1, 2, \dots, 10$ .
- We consider a generalised CGMY process (with a diffusion component) time-changed by a Heston (1993) activity rate (parameters from calibration to the market prices of vanilla options on the S & P500 stock index).
- To see effect of drift and correlation:
- We consider two possible choices, labelled (a) and (b) for the values of the interest-rate  $r(t)$  and the dividend yield  $q(t)$ . In the first choice (a),  $r(t) = 0$ ,  $q(t) = 0$ , for all  $t$ . In the second choice (b),  $r(t) = 0.065$ ,  $q(t) = 0.015$ , for all  $t$ .
- We consider three different combinations for the correlation  $\rho$  between the activity rate and the diffusion component of the CGMY process: Namely,  $\rho = -0.99$ ,  $\rho = 0$  and  $\rho = 0.99$ .





- Now let's consider the problem of estimating process parameters from historical data.
- Either assume that we have structure-preserving risk-premia which means we have time-changed Lévy process dynamics under the real-world physical measure  $\mathbb{P}$  and under  $\mathbb{Q}$  (with, in general, different parameters).
- Or simply regard estimating process parameters as a separate problem.
- Either way, we assume henceforth time-changed Lévy process dynamics under  $\mathbb{P}$ .

- Suppose we are given stock prices  $S(t_j)$  for times  $t_j$ , for  $j = 1, 2, \dots, N$  where  $N$  is large and is of the form  $N = LM$ , for integers  $L$  and  $M$ .
- Let us identify  $L$  and  $M$  as follows:  $L$  is the total number of days on which we observe the stock prices and on each day we observe  $M$  prices (not necessarily at equal intervals).
- The quadratic variation  $QV(\ell)$  of log of the stock price over the period from time  $t_{(\ell-1)M}$  to time  $t_{\ell M}$  (i.e. on the  $\ell^{\text{th}}$  day) is defined as:

$$QV(\ell) \equiv \lim_{\hat{N} \rightarrow \infty} \sum_{n=1}^{\hat{N}} (\log(S(u_n)/S(u_{n-1})))^2,$$

for any sequence of partitions  $t_{(\ell-1)M} \equiv u_0 < u_1 < u_2 < \dots < u_{\hat{N}-1} < u_{\hat{N}} \equiv t_{\ell M}$  with  $\sup\{u_n - u_{n-1}\} \rightarrow 0$ .

- Note that on the  $\ell^{\text{th}}$  day, for each  $\ell = 1, 2, \dots, L$ , we can compute the realised variance  $\tilde{R}V(\ell, M)$  via:

$$\tilde{R}V(\ell, M) \equiv \sum_{m=1}^M (\log(S(t_{(\ell-1)M+m})/S(t_{(\ell-1)M+m-1})))^2.$$

- This (discrete) realised variance  $\tilde{R}V(\ell, M)$  is clearly a discrete approximation to the quadratic variation  $QV(\ell)$ .
- There is a central limit theorem type result (Barndorff-Nielsen and Shephard (2004)) that says that  $\tilde{R}V(\ell, M)$ , for each  $\ell = 1, \dots, L$ , are (approximately) multi-variate normal provided  $M$  is not too small (say,  $M \geq 15$ ).
- Recall  $M$  is the number of observations per day of the stock price.

- High-frequency data uses values of  $M$  that are large, for example,  $M$  equal to 288 (every 5 minutes, 24 hours in a working day - Barndorff-Nielsen and Shephard (2004)) or sampling every 60 seconds ( $M = 480$  for 8 hour working day) or every 10 seconds ( $M = 2880$  for 8 hour working day) - Barndorff-Nielsen, Hanson, Lunde and Shephard (2008).
- Good point: Large  $M$  seems to use more data - therefore better estimates??
- Bad point: Concern that market microstructure effects - eg minimum tick-size, indicative prices or actual transactions prices, illiquidity - distort the estimates if  $M$  is too large.

- Existing papers have attempted to estimate the model parameters via maximum likelihood with a log-likelihood function based on multi-variate normal. For this we need to have expressions for the following quantities:  $\mathbb{E}_{t_0}^{\mathbb{P}}[\tilde{R}\tilde{V}(\ell, M)]$ ,  $\text{Var}_{t_0}^{\mathbb{P}}[\tilde{R}\tilde{V}(\ell, M)]$  and  $\text{Cov}_{t_0}^{\mathbb{P}}[\tilde{R}\tilde{V}(\ell, M), \tilde{R}\tilde{V}(j, M)]$  for all  $j$  and all  $\ell$ .
- In trying to compute these quantities, existing papers seem to make at least one assumption out of the following:
  - (1) Assume continuously monitored (ie actually use the expressions for  $\mathbb{E}_{t_0}^{\mathbb{P}}[\tilde{R}\tilde{V}(\ell, \infty)]$ , etc);
  - (2) Ignore drift;
  - (3) Assume independence between  $X_t$  and  $Y_t$ ;
  - (4) Assume continuous sample paths (i.e.  $X_t$  is actually Brownian motion).
- We can compute these quantities without making any of these assumptions.



- Can compute  $\mathbb{E}_{t_0}^{\mathbb{P}}[\tilde{R}V(\ell, M)]$  exactly for finite  $M$  using equation (1) (the formula for the price of a discretely monitored variance swap).
- Can compute  $\text{Var}_{t_0}^{\mathbb{P}}[\tilde{R}V(\ell, M)]$  via the fourth derivative of the joint extended characteristic function  $\Phi(z; j)$ .
- Can compute  $\text{Cov}_{t_0}^{\mathbb{P}}[\tilde{R}V(\ell, M), \tilde{R}V(j, M)]$  for all  $j$  and all  $\ell$  by considering an extended characteristic function of the form  $\mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(iz_1 \log \frac{S(t_j)}{S(t_{j-1})} + iz_2 \log \frac{S(t_k)}{S(t_{k-1})})]$
- But how much difference does it make (compared to making the four assumptions on the last slide: (1) Continuously monitored; (2) Ignore drift; (3) Independence between  $X_t$  and  $Y_t$ ; (4) Continuous sample paths)?

- Used same CGMY data as before. Values of the “C” parameters were normalised so that  $\mathbb{E}_{t_0}^{\mathbb{P}}[\tilde{R}\tilde{V}(\ell, \infty)] = 0.25$ , exactly.
- $\mathbb{E}_{t_0}^{\mathbb{P}}[\tilde{R}\tilde{V}(\ell, M)]$  expressed as an annualised volatility equivalent for different values of  $M$ .

$\mathbb{E}_{t_0}^{\mathbb{P}}[\tilde{R}\tilde{V}(\ell, M)]$	$M$
0.250206	1
0.250103	2
0.250051	4
0.250025	8
0.250012	16
0.250006	32
0.250002	64
0.250001	128

- Answer: It makes little difference. The theoretical value of  $\mathbb{E}_{t_0}^{\mathbb{P}}[\tilde{R}V(\ell, M)]$  is very, very insensitive to  $M$ .
- We saw that the drift and the correlation between  $X_t$  and  $Y_t$  make very little difference to the price of a variance swap with, for example, daily monitoring. Its the same story with high-frequency data.
- Conclusion:
  - (1) Can assume continuously monitored (ie actually use the expressions for  $\mathbb{E}_{t_0}^{\mathbb{P}}[\tilde{R}V(\ell, \infty)]$ , etc);
  - (2) Can ignore drift;
  - (3) Little to be lost (for this estimation method) by assuming independence between  $X_t$  and  $Y_t$  (because this estimation method cannot produce reliable non-zero estimates).
- If worried about market microstructure effects, one can safely use a smaller value of  $M$  (for say  $M \geq 15$ ) - or even better (Ait-Sahalia (2005)), model market microstructure effects explicitly and find the optimal choice of  $M$  based on trading off more data against microstructure noise - not based on discrete monitoring effects.

- We **cannot** ignore jumps. We can show approximately

$$\text{Var}_{t_0}^{\mathbb{P}}[\tilde{RV}(\ell, M)] \approx \frac{-\overline{\psi}_X''''(0)}{365} + (\overline{\psi}_X''(0))^2 \text{Var}_{t_0}^{\mathbb{P}}[Y_{\ell M+m-1} - Y_{(\ell-1)M+m-1}].$$

- The second term will be very small (eg  $10^{-10}$  or  $10^{-11}$ ) for realistic data. The first term (excess kurtosis) would be identically equal to zero for Brownian motion. In practice (based on high-frequency foreign exchange data in Barndorff-Nielsen and Shephard (2004)), the first term is of the order of one million times bigger than the second term. The Barndorff-Nielsen and Shephard data implies a value of  $-\overline{\psi}_X''''(0)$  which is of the order of 0.08 to 0.8 (my CGMY data implies a value of 0.092 which is in the right ballpark).

- Generally speaking, discrete monitoring makes little difference to the prices of variance swaps and proportional variance swaps (in the paper, we show more or less the same story for self-quantoeed variance swaps, gamma swaps and skewness swaps). This means they are also little affected by the value of  $\text{Correl}(X_t, Y_t)$ .
- Jumps in the underlying dynamics make a lot of difference (there are more examples in the paper) - this is especially true with asymmetric jumps.
- This motivates empirical studies which try to determine how much of the negative skewness seen in stock price returns (under  $\mathbb{P}$  and  $\mathbb{Q}$ ) comes from a negatively skewed Lévy process and how much comes from a negative value of  $\text{Correl}(X_t, Y_t)$  (the maximum likelihood method outlined earlier seems incapable of doing this).
- The paper (“Variance derivatives: Pricing and convergence”) on which this talk is based will soon be on my website:  
<http://www.john-crosby.co.uk> . (or email me - address on website).

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